# Towards Understanding the Min-Sum Message Passing Algorithm for the Minimum Weighted Vertex Cover Problem: An Analytical Approach

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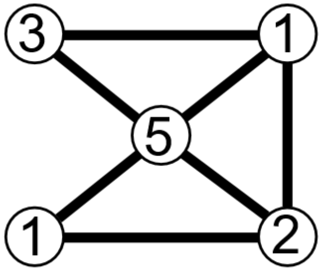
# Summary

- We constructed an analytical framework to study the min-sum message passing algorithm applied to minimum weighted vertex cover problems.
- Our framework correctly predicts the asymptotic behavior of the algorithm applied to minimum weighted vertex cover problem with single loop.
- Step toward analytical understanding of message passing algorithm.

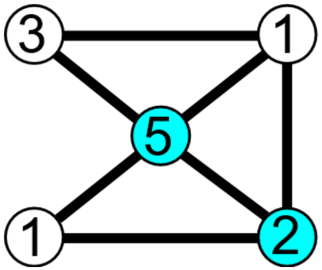
# Contents

# Minimum Weighted Vertex Cover (MWVC) Problems

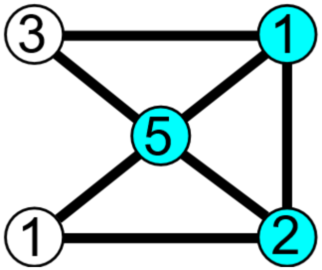
Vertex Cover (VC):
Subset $U$ of vertices such that every edge is incident on some vertex in $U$.



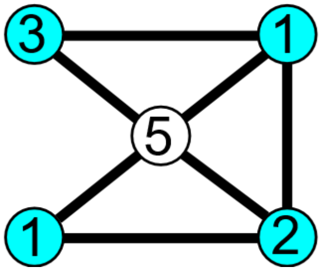

Minimum Weighted Vertex Cover:
A vertex cover whose total weight is minimum.



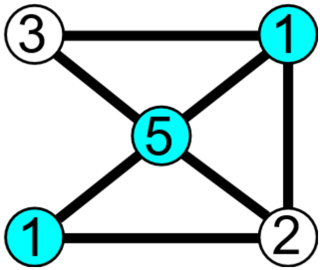

☒ VC
☒ MWVC




☑ VC
☒ MWVC




☑ VC
☑ MWVC




☑ VC
☑ MWVC

# Minimum Weighted Vertex Cover (MWVC) Problems

- NP-Hard
- Appear in problems such as auction problem (Sandholm 2002), kidney exchange, error correcting code (McCreesh et al. 2017).
- Weighted constraint satisfaction problems, which are the most general form of combinatorial optimization problems, can be reduced to MWVC problems (Xu et al. 2017)
- Efficient approximation methods for MWVC have large impact

# Min-Sum Message Passing (MSMP) Algorithm

- MSMP is a variant of belief propagation method
- Widely used as estimate for combinatorial optimization problems which avoid exponential time complexity (Yediddia et al. 2003)
- Application to probabilistic reasoning, AI, statistical physics, etc. (Mezard and Montanari 2009, Yedidia et al. 2003)
- Iterative method which converges and is correct for trees, but not fully understood for loopy graphs (Mezard and Montanari 2009)

# MSMP Applied to MWVC Problems

- Weigt and Zhou 2008 studied message passing for minimum vertex cover
- Sanghavi et al. 2008 studied the correctness of max-product message passing algorithm for maximum weighted independent set (equivalent to MWVC)
- Little analytical work for MSMP for MWVC with random graph

# MSMP Applied to MWVC Problems

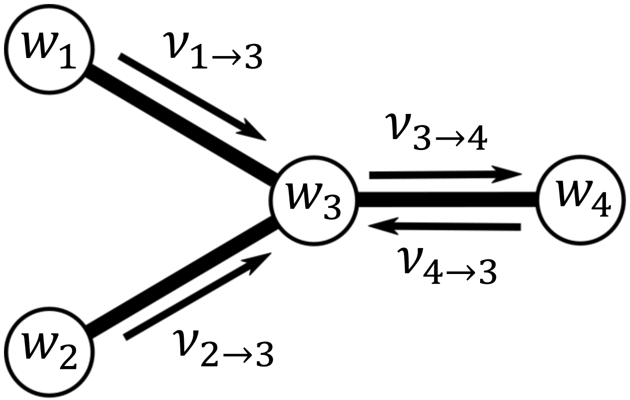
- Let $v_{i \to j}$ denote the message from $i$ to $j$
- Initialize $v_{i \to j} = 0$ for all messages
- Update the messages as follows

$$v_{i \to j} = max\left\{0,\ w_i - \sum_{k \in N(i) \setminus j} v_{k \to i}\right\} \quad (0 \le v_{i \to j} \le w_i)$$

- After the messages converge, choose vertex $i$ if

$$w_i \le \sum_{k \in N(i)} v_{k \to i}$$

$$v_{3 \to 4} = max\{0, w_3 - (v_{1 \to 3} + v_{2 \to 3})\}$$



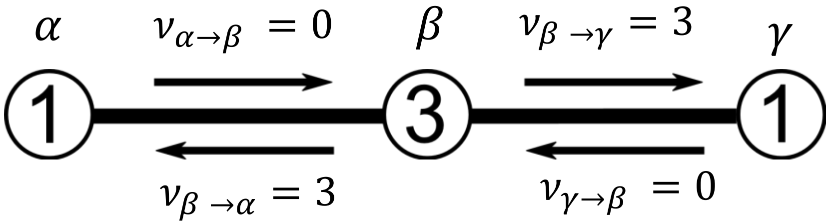

$$w_3 \le v_{1 \to 3} + v_{2 \to 3} + v_{4 \to 3}$$

# MSMP Applied to MWVC Problems

- $v_{i \to j}$ is a "warning to cover" from $i$ to $j$
- For vertex $\alpha$:

$$\sum_{k \in N(\alpha)} v_{k \to \alpha} = v_{\beta \to \alpha} = 3 \geq w_\alpha$$

$\Rightarrow$Select vertex $\alpha$ for MWVC




- For vertex $\beta$:

$$\sum_{k \in N(\beta)} v_{k \to \beta} = v_{\alpha \to \beta} + v_{\gamma \to \beta} = 0 < w_\beta$$

$\Rightarrow$Do not select vertex $\beta$

$$v_{i \to j} = max \left\{ 0,\ w_i - \sum_{k \in N(i) \setminus j} v_{k \to i} \right\}$$

# Probability Distribution of Messages

- Consider a MWVC problem with random graph with vertex weight distribution $g(w)$
- Assume: upon convergence the probability distribution of $v_{i\to j}$ only depends on $w_i$
- $F(v_{i\to j}; w_i)$ : Cumulative probability of vertex with $w_i$ sending message up to $v_{i\to j}$
- $f(v_{i\to j}, w_i) = \frac{\partial F(v_{i\to j}; w_i)}{\partial v_{i\to j}}$: Probability density of vertex with $w_i$ sending message $v_{i\to j}$
- $\int_0^{w_i} f(v_{i\to j}; w_i)\, dv_{i\to j} = 1$: Normalization condition (since $0 \le v_{i\to j} \le w_i$)

# Probability Distribution of Messages

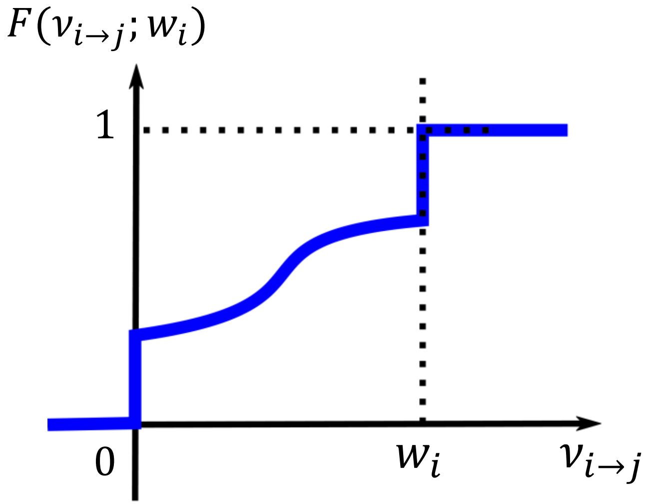
$$F(\nu_{i\to j};w_i) = \Theta(\nu_{i\to j})P(0;w_i) + F_m(\nu_{i\to j};w_i) + \Theta(\nu_{i\to j}-w_i)P(w_i;w_i),$$

$$f(\nu_{i\to j};w_i) = \delta(\nu_{i\to j})P(0;w_i) + f_m(\nu_{i\to j};w_i) + \delta(\nu_{i\to j}-w_i)P(w_i;w_i),$$

$$\nu_{i\to j} = max\left\{0, w_i - \sum_{k\in N(i)\setminus j} \nu_{k\to i}\right\}$$

$$(0 \le \nu_{i\to j} \le w_i)$$

- $P(0;w_i)$: Probability of vertex with $w_i$ sending message 0
- $F_m(\nu_{i\to j};w_i)$: Smooth function for $0 < \nu_{i\to j} < w_i$
- $P(w_i;w_i)$: Probability of vertex with $w_i$ sending message $w_i$



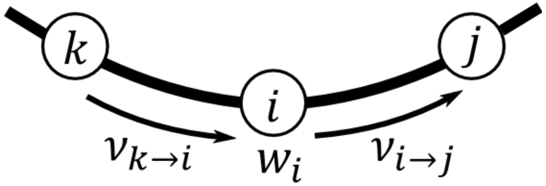
# MWVC with Infinite Single Loop

- Single loop with weight distribution $g(w)$
- $\nu_{i\to j} = max\{0, w_i - \nu_{k\to i}\}$

$$f_m(\nu_{i\to j}; w_i) = \int_{(w_i - \nu_{i\to j})^-}^{+\infty} \mathrm{d}w_k\, g(w_k) f(w_i - \nu_{i\to j}; w_k)$$

$$P(0; w_i) = \int_{w_i^-}^{+\infty} \mathrm{d}w_k\, g(w_k) \int_{w_i^-}^{w_k} \mathrm{d}\nu_{k\to i}\, f(\nu_{k\to i}; w_k)$$

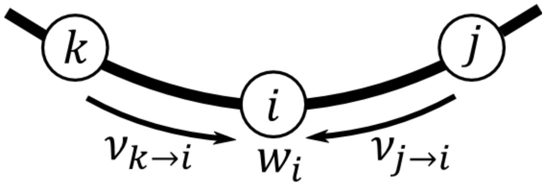
$$P(w_i; w_i) = \int_{0^-}^{+\infty} \mathrm{d}w_k\, g(w_k) P(0; w_k)$$



# MWVC with Infinite Single Loop

- Include vertex $i$ if $w_i \le \nu_{j\to i} + \nu_{k\to i}$
- $\bar{w}$: Average contribution per vertex to total weight of MWVC ($^{Total\ Weight\ of\ MWVC}/_{N}$ in discrete case)

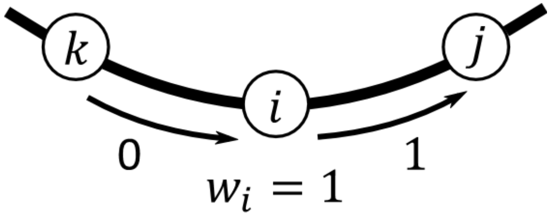
$$\bar{w} = \int_{0-}^{+\infty} \mathrm{d}w_j\, g(w_j) \int_{0-}^{+\infty} \mathrm{d}w_k\, g(w_k)$$
$$\times \int_{0-}^{w_j} \mathrm{d}\nu_{j\to i}\, f(\nu_{j\to i}; w_j) \int_{0-}^{w_k} \mathrm{d}\nu_{k\to i}\, f(\nu_{k\to i}; w_k)$$
$$\times \int_{0-}^{\nu_{j\to i}+\nu_{k\to i}} \mathrm{d}w_i\, w_i g(w_i),$$



# MWVC with Infinite Single Loop – Constant Weight

- $g(w) = \delta(w - 1)$ (equivalent to MVC)
- Solution:

$$f(\nu_{i\to j}; 1) = \frac{1}{2}\left[\delta(\nu_{i\to j} - 1) + \delta(\nu_{i\to j} - 0)\right]$$



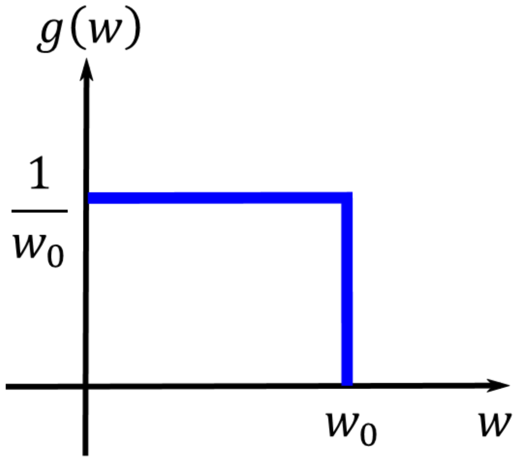

- Every message is either 0 or 1 with probability 0.5
- Same as the result of MVC problem

# MWVC with Infinite Single Loop – Uniform Weight Distribution

$$g(w) = \frac{1}{w_0}\Theta(w)\Theta(w_0 - w)$$

- Integral equations were converted to differential equations
- Key to solve the problem: Linear idempotent differential equation (Falbo 2003)

$$\bar{w} = \frac{1 + \sin(1) - 2\cos(1)}{2 + 2\sin(1)} w_0 \approx 0.2066 w_0$$



# Numerical Experiment – Uniform Weight Distribution

- $w_0 = 1$ (prediction: $\overline{w} = 0.2066$ as $N \to \infty$)
- Choose 16 values of $N$ from 20 to $10^5$
- Create 50 instances of MWVC problem with single loop with uniform distribution for each $N$
- Run MSMP for MWVC and compute $\overline{w}$ over 50 instances for each $N$

$$\overline{w} = \frac{Total\ Weight\ of\ VC}{N}$$

- Run dynamic programming for optimal solution of $\overline{w}$
- Compare the results to the analytical prediction of $\overline{w}$

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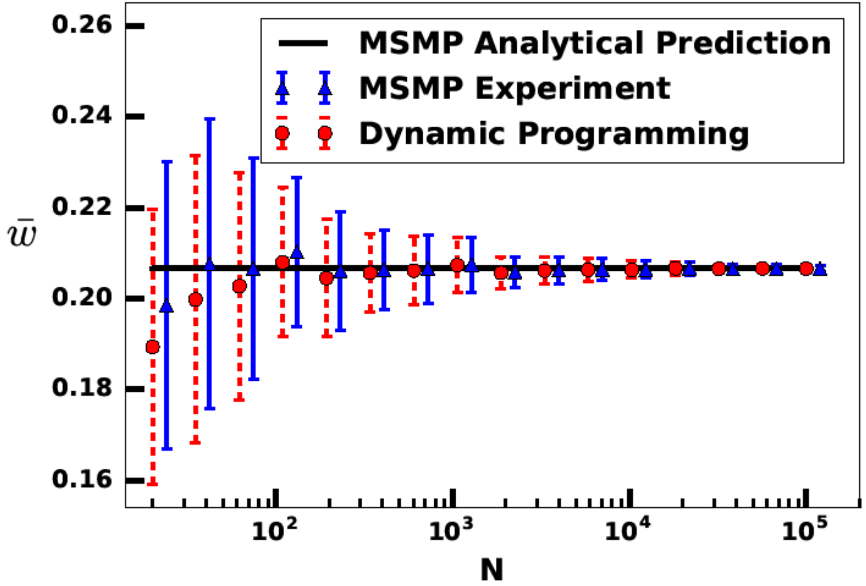
# Numerical Experiment – Uniform Weight Distribution

- Prediction: $\bar{w} = 0.2066$ as $N \to \infty$
- MSMP algorithm matches with exact solution as $N \to \infty$
- Correctly predicts asymptotic behavior of MSMP algorithm
- Correctly predicts the solution to MWVC problem for large $N$



# Conclusions and Future Work

- Developed an analytical framework for MSMP for MWVC problems
- Analyzed MWVC problems with single loop with uniform weight
- Correctly predicted the asymptotic behavior of MSMP algorithm
- Correctly predicted the solution to MWVC of single loop with large $N$
- Supports the use of MSMP for MWVC
- Step toward understanding of MSMP algorithm on loopy graphs

- Analysis on other weight distribution (e.g. exponential)
- Analysis on more general loopy graphs

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